**Chapter 8 Sinusoidal Steady State**

The present chapter is concerned with introducing a special methodology for the analysis of a circuit excited by sinusoidal functions of time in the steady state. The methodology is based on expressing voltages and currents as phasors and the voltage-current relation of circuit elements in terms of impedance. Linear differential equations describing the behavior of circuits involving capacitors and inductors are thereby reduced to algebraic equations involving the imaginary unit *j*. This greatly simplifies circuit analysis and extends to the sinusoidal steady state all the concepts, theorems, and procedures applied to resistive circuits under dc conditions, in addition to other advantages mentioned in Section 8.1.

The chapter is therefore mainly concerned with developing the concepts of phasors and impedance and applying them in several illustrative examples.

**8.1The Sinusoidal Function**

A voltage or a current that varies sinusoidally with time can be represented as:



(8.1.1)

where *Ym* is the amplitude, *ω* is the angular frequency, and *θ* is the phase angle (Figure 8.1.1). The time interval between successive repetitions of the same value of *y* is the **period** *T*. The full range of values of the function over a period is a **cycle**. The frequency *f* of repetitions of the function is:

 (8.1.2)

where *T* is in seconds, *f* is in cycles per second, or hertz (Hz), and *ω* is in rad/s. Voltages and currents that vary sinusoidally with time are designated as **ac quantities**, where ac stands for ‘alternating current’.

As *ω* decreases, *T* increases (Equation 8.1.2), the sinusoid becoming “flatter”. If *ω* = 0, the sinusoid of Figure 1.1.1 becomes a dc quantity of magnitude *Ym*.

The sinusoidal function is a periodic function, that is, a function consisting of cycles that repeat at a given frequency. However, it is the only periodic function that has only a single frequency. In fact, according to Fourier’s theorem (Chapter 16), a periodic function can generally be considered as the sum of a theoretically infinite number of cosine and sine functions whose frequencies are integral multiples of the frequency of the periodic function.

An important implication of this characteristic of sinusoidal functions is:

**Concept: *When a sinusoidal excitation is applied to an LTI circuit,* all *the currents and voltages in the circuit are sinusoidal functions, generally having different amplitudes and phase angles, but all having the same frequency as the excitation*.**

The reason for this is that voltages and currents in an LTI circuit are subject to linear operations such as: i) scaling, or multiplication by a constant, as in voltage or current division, ii) addition and subtraction, as in KCL or KVL, and iii) differentiation and integration, as in the *v*-*i* relations of inductors and capacitors. When sinusoidal functions are subjected to any of these linear operations, they remain sinusoidal functions of the same frequency. The amplitudes and phase angles of the resulting sinusoidal functions will differ from those of the original functions, but no new frequencies are produced.

Another important property of the sinusoidal function, which is used in this chapter, is embodied in De Moivre’s theorem:

 (8.1.3)

where *j* =  is the *imaginary unit* and *Ym* is the amplitude of the sinusoidal function.  is a complex quantity, whose *real component* is *Ym*cos*ωt* and whose *imaginary component* is *jYm*sin*ωt*. The term *Ym*sin*ωt*, without *j*, is the *imaginary part*. The real part and the real component are, of course, one and the same.



Equation 8.1.3 can be plotted in the complex plane, or Argand diagram, where the real part is plotted along the horizontal, or real, axis, and the imaginary part is plotted along the vertical, or imaginary, axis (Figure 8.1.2).  is plotted in *polar form*, as a position vector OP , that is a

vector drawn from the origin, whose magnitude is *Ym* and whose angle with respect to the horizontal axis is *ωt*. This angle increases with time, so that the vector OP rotates counterclockwise about the origin at an angular velocity *ω*. In *rectangular form*, *Ym*cos*ωt* is the projection of  along the real axis, and *Ym*sin*ωt* is its projection on the imaginary axis. As time progresses, the lengths of these projections vary sinusoidally with time, as illustrated by the dashed extensions in Figure 8.1.2.

Analysis of the sinusoidal steady state is the derivation of the currents and voltages in a circuit after a sinusoidal excitation has been applied for a sufficiently long time. When any excitation is suddenly applied to a circuit, there is generally an initial “transient”, or temporary, response that dies out with time, as explained in later chapters. After this transient dies out, a steady-state remains, in which the currents and voltages repeat according to some pattern that depends on the time variation of the excitation.

Analysis of the sinusoidal steady state is of fundamental importance for the following reasons:

1. The sinusoidal steady state is used in several practical cases, as in the frequency response of systems (Chapter 14) and in power systems analysis (Chapter 25). Electric power is efficiently generated, transmitted, and distributed as sinusoidal voltages.
2. The methodology used in steady state sinusoidal analysis is based on expressing sinusoidal voltages and currents as phasors and the *v-i* relations in terms of impedance. This is extremely convenient in that it extends to the sinusoidal steady state all the concepts, theorems, and procedures that apply to resistive circuits under dc conditions.
3. This representation of sinusoidal voltages and currents as phasors and the *v-i* relations in terms of impedance are later applied to the analysis of frequency responses of circuits (Chapter 14), the responses to periodic inputs (Chapter 16), complex power and three-phase systems (Chapter 25), and the Fourier transform (Chapter 22). Moreover, the same representation can be readily generalized to the most powerful method for analyzing linear systems, namely that of the Laplace transform (Chapter 21).

**8.2 Phasors**

***Responses to Sinusoidal Excitations***

Consider the circuit of Figure 8.2.1a in which a current source *iSRC* = *Im*cos*ωt* A is applied to a series *RL* circuit. It is desired to determine the voltage *v* across the

series combination. Note that a tilde symbol is added to the source to denote a steady sinusoidal excitaion.



From KVL, *v = vR* + *vL*, where *vR* = *RiSRC* and *vL* = *LdiSRC*/*dt*. KVL becomes:

 (8.2.1)

Substituting, *iSRC* = *Im*cos*ωt*,

 (8.2.2)

To combine the sine and cosine terms into a single function, the RHS is multiplied and divided by  to give:

 (8.2.3)

We now define an angle whose cosine is  and sine is  (Figure 8.2.1b). Equation 8.2.3 becomes:



 (8.2.4)

It is seen that this procedure for obtaining *v* is rather complicated, even in this simple case, as it involves writing KVL using a time derivative, followed by manipulation of trigonometric functions. In contrast, *V* in a similar dc case (Figure 8.2.2) can be written from Ohm’s law as:



 (8.2.5)

A pertinent question, therefore, is how to reduce the derivation of voltage in Figure 8.2.1a to be much like that in Figure 8.2.2.

The key to achieving this goal is to consider the excitation to be  rather than *Im*cos*ωt*. Then, , and . Note that differentiation has been reduced simply to multiplication by *jω*. KVL in Figure 8.2.1 gives:

 (8.2.6)

This is a step in the right direction, since Equation 8.2.6 is similar in form to

Equation 8.2.5 in that a constant, but complex quantity (*R* + *jωL*), multiplies the current. Equation 8.2.6 will be generalized shortly to look more like Equation 8.2.5. But before doing so, it may be noted that Equation 8.2.6 is the response to  and not to *Im*cos*ωt*, as in Equation 8.2.4, as required. Recall, however, that

 (8.2.7)

in accordance with Equation 8.1.3. The response to *Im*cos*ωt* was determined earlier (Equation 8.2.3), so let us determine the response to *iSRC* = *Im*sin*ωt.* Equation 8.2.1 becomes in this case:

 (8.2.8)

Multiplying and dividing the RHS by  gives





or,  (8.2.9)

where *θ* = tan-1*ωL*/*R*, as in Figure 8.2.1b. By superposition, the response to  is the sum of the RHS of Equation 8.2.4 and the RHS of Equation 8.2.9 multiplied by *j*. That is,





 (8.2.10)

where  is the magnitude of the voltage response. But Equation 8.2.6 is identical to Equation 8.2.10. (*R* + *jωL*) can be expressed in polar form as  where *θ* = tan-1(*ωL*/*R*), as in Figure 8.2.1b. The expression for *v* becomes:



As in Equation 8.2.10. What we have shown, therefore, is that the response to the complex sinusoidal excitation  gives, as its real part, which is , the response to *Im*cos*ωt*. It also gives, as its imaginary part, which is , the response to *Im*sin*ωt* (Equation 8.2.4). In other words, applying the excitation  is equivalent to applying the two

excitations *iSRC* = *Im*cos*ωt* and *iSRC* = *Im*sin*ωt* simultaneously but with the response being the complex sum of the responses to each excitation applied alone. This underlies an important concept, namely,

**Concept *When a complex sinusoidal excitation Kejωt is applied to an LTI circuit, the response is a complex sinusoidal function whose real part is the response to the real part of the excitation, Kcosωt, applied alone, and whose imaginary part is the response to the imaginary part of the excitation, Ksinωt, applied alone.***

The reason for this is that in linear operations, the real and imaginary parts of a complex quantity do not mix; they retain their respective identities. Thus, if *x* = (*a* + *jb*) and *y* = (*c* + *jd*), *x* + *y* = (*a* + *b*) + *j*(*c* + *d*), where the real part of (*x* + *y*) is the sum of the two real parts (*a* + *b*), and the imaginary part is the sum of the two imaginary parts (*c* + *d*). On the other hand, multiplying *x* and *y*, which is a nonlinear operation, gives *xy* = (*ab* – *cd*) + *j*(*bc* + *ad*), where the real and imaginary parts of *xy* and a mixture of the real and imaginary parts of *x* and *y*.

It follows that if the response to an excitation *K*cos*ωt* or *K*sin*ωt* is required, a complex sinusoidal excitation can be applied and the real and imaginary parts of the response will be the responses to *K*cos*ωt* and *K*sin*ωt*, respectively.

The next step in our quest to reduce Equation 8.2.6 to the form of Equation 8.2.5 is to omit the time variation  from the RHS of Equation 8.2.6. The justification, as explained earlier, is that when an LTI circuit is excited at a single frequency, all the currents and voltages in the circuit will be sinusoids having the same given frequency *ω*, but with different amplitudes and phase angles. No information is lost by omitting the time variation . This leads to phasor notation, as will be elaborated next.

***Phasor Notation***

The complex excitation  and the complex response  (Equation 8.2.10) can be plotted as position vectors on an Argand diagram (Figure 8.2.3a). Both vectors rotate counterclockwise at an angular speed *ω*, retaining the phase angle *θ* between them as they do so. The angular speed is the same, because as emphasized earlier, all the voltage and current responses in an LTI circuit have the same frequency as the excitation. The information of interest in Figure 8.2.3a is the magnitude of the various vectors and their relative phase angle. It follows that no significant information is lost by freezing the rotation of the vectors at a particular

value of *t*, which can



be conveniently taken as *t* = 0.  becomes ,

and  becomes , as in Figure 8.2.3b. These complex quantities are referred to as phasors. Thus,

**Definition *A phasor is a quantity such as Vmejθ***

***representing a complex sinusoidal function of time, but with the time variation suppressed.***

Phasors are written in boldface and expressed as a magnitude and phase angle:

**V** (8.2.11)

It should be emphasized that phasors drawn in a given Argand diagram are implicitly complex sinusoidal functions of time rotating at the same angular frequency. Phasors that are implicitly rotating at different frequencies cannot, and should not, be dawn on the same Argand diagram. Nor should complex quantities, such as (*R* + *jωL*) (Equation 8.2.6), be drawn on the same Argand diagram as phasors (Figure 8.2.3c).

***Properties of Phasors***

Being geometrically position vectors, phasors have magnitude and direction, just like vectors, but with the additional property that their horizontal component is a real quantity, and their vertical component is



an imaginary quantity. They can therefore be scaled, added and subtracted like vectors.

Thus, multiplying a phasor by a real quantity *K* multiplies its magnitude by *K,* without changing its phase angle. Two phasors **Y1** and **Y2** can be added by drawing **Y2** such that its origin lies at the tip of **Y1** (Figure 8.2.4a). The sum **Y1** + **Y2** is the phasor whose origin is that of **Y1** and whose tip is that of **Y2**. The sum of **Y1** and **Y2** may be also obtained by applying the ‘parallelogram rule’, as in Figure 8.2.4a. The real part of **Y1** + **Y2** is: *Y*1cos*θ*1+*Y*2cos*θ*2, and its imaginary part is: *Y*1sin*θ*1+*Y*2sin*θ*2. It follows thatthe magnitude of **Y1** + **Y2** is:

|**Y1** + **Y2|** (8.2.12)

The phase angle of **Y1** + **Y2** is:

∠(**Y1** + **Y2)** (8.2.13)

The phasor difference **Y1** – **Y2** is obtained by adding **Y1** and -**Y2**, where -**Y2** is a phasor of the same magnitude as **Y2** but having a phase angle (*θ2* + *π*) (Figure 8.2.4b). Alternatively, the phasor **Y1** – **Y2** may be obtained as the phasor whose origin lies at the tip of **Y2** and whose tip lies at the tip of **Y1**. Then: **Y1** = **Y2** + (**Y1** – **Y2**).



A phasor **Y** (Figure 8.2.5a) may be multiplied by a complex quantity  (Figure 8.2.5b):

 (8.2.14)

The product is a phasor of magnitude *AY* and phase angle (*θ* +*α*) (Figure 8.25a).

A phasor **Y** may be divided by a complex quantity :

 (8.2.15)

The quotient is a phasor of magnitude  and phase angle (*θ* –*α*) (Figure 8.2.5c).

A special case is multiplication and division by *j*. In the complex plane, *j* is an imaginary quantity of unit magnitude and a phase angle of *π*/2:

 (8.2.16)

Multiplying a phasor by *j* rotates the phasor through an angle *π*/2 counterclockwise without changing its magnitude (Figure 8.2.6a). Dividing a phasor by *j*, or conversely multiplying it by -*j*, since , rotates the phasor through an angle *π*/2 clockwise without changing its magnitude (Figure 8.2.6b).



**Exercise 8.2.1**

Determine the product and quotient of  and  by working in rectangular coordinates and in polar coordinates.

Ans. Product:  Quotient: 

**Example 8.2.1**

It is required to determine the magnitude, phase, real, and imaginary parts of , where *a*, *b*, *c*, and *d* are positive constants.

***Solution*:** Let us rationalize *Y*, that is, make its denominator real, by multiplying numerator and denominator by the complex conjugate of the denominator, *c – jd*. Thus:

 (8.2.17)

The real part of *Y* is  its imaginary part is.

The magnitude of *Y* may be obtained as the square root of the sum of the squares of the real and imaginary parts. An easier way is to convert the numerator and denominator to polar coordinates. Thus: .

The magnitude of *Y* is therefore . The phase angle of *Y* is (*θ*1 – *θ*2) =

tan-1(*b*/*a*) – tan-1(*d*/*c*), that is, the phase angle is of the phase angle numerator minus that of the denominator. The following should be noted:

1. Whereas the magnitude of *Y* is the magnitude of the numerator divided by that of the denominator, the real part of *Y* is *not* the real part of the numerator divided by that of the denominator. Nor is the imaginary part of *Y* the imaginary part of the numerator divided by that of the denominator.
2. In determining the phase angle from the real and imaginary parts, the actual signs of these parts must be retained without change, as illustrated in Figure 8.2.7. Otherwise, the angle will be incorrect. Thus, assuming *a* and *b* are positive constants, tan-1(*b*/*a*) is an angle *θ* in the first quadrant. However, tan-1(-*b*/-*a*) is the ratio of two negative components, -*b* to -*a*, so that the angle is (±*π* + θ) in the third quadrant. Similarly, tan-1(-*b*/*a*) is an angle -*θ* in the fourth quadrant, whereas tan-1(*b*/-*a*) is an angle (*π* – *θ*) in the second quadrant.



**Problem-Solving Tip**

* Complex quantities are conveniently added or subtracted in rectangular form, and are conveniently multiplied or divided in polar form.

**Exercise 8.2.2**

Show that the square root of the sum of the squares of the real and imaginary parts of the RHS of Equation 8.2.17 reduces to 

**8.3 Phasor Relations of Circuit Elements**

***Phasor Relations for a Resistor***

If the current through a resistor is  A, the voltage across the resistor is

 V. In phasor notation **I** A, and **V** V, (Figure 8.3.1a), or:



**V****I**, or **I****V** (8.3.1)

According to the interpretation of complex sinusoidal excitation, a current *Im*cos*ωt* A produces a voltage *RIm*cos*ωt* V, and a current *Im*sin*ωt* V produces a voltage *RIm*sin*ωt* V. The voltage and current are in phase (Figure 8.3.1b).

If *i* = *Im*cos*ωt* and *v* = *RIm*cos*ωt*, the instantaneous power absorbed and dissipated by the resistor, based on the assigned positive directions of Figure 8.3.1a, is:

 (8.3.2)

*p,* plotted in Figure 8.3.1b, is never negative, because a resistor does not deliver power. From Equation 8.3.2 *p* has an average component of  and a sinusoidal component of zero average, amplitude  and frequency 2*ω*. Formally, the average power *P* is obtained by integrating *p* over a period, which gives the energy dissipated over the period, and dividing by the period:

 (8.3.3)

Note that  is the mean square of *i* = *Im*cos*ωt*, over a period, since,

 (8.3.4)

The square root of this mean is the **root-mean-square**, or rms value of *i*, which is denoted as *I*rms. Thus

 (8.3.5)

Substituting in Equation 8.3.3,

 (8.3.6)

Had we considered *i* = *Im*sin*ωt* instead of *i* = *Im*cos*ωt*, only the sign of the cos2*ωt* changes in preceding equations, but since the integral of this term averages

to zero over the period, *P* and *Irms* are unchanged. Moreover, if we consider *v* = *Vm*cos*ωt*, where *Vm* = *RIm*, the same procedure yields:

, ,  (8.3.7)

It follows from Equation 8.3.3 that

 (8.3.8)

Since the power dissipated in *R* by a dc current *I* is , it follows that:

***Concept*** ***A current of rms value Irms, or a voltage of rms value Vrms, dissipate the same power in a given resistor as a dc current, or a dc voltage, of the same value*.**

Using rms values of sinusoidal currents and voltages for power calculations results in expressions of the same form as under dc conditions. The rms value is also known as the **effective** value.

***Phasor Relations for a Capacitor***

If the current through a capacitor is  A, the voltage across the capacitor is:

 (8.3.9)

where *K* is the constant of integration that shifts the voltage in the vertical direction and defines its average value. When a sinusoidal current of zero average is applied to an uncharged capacitor, the resulting steady-state voltage is sinusoidal and of zero average, so that *K* = 0.

In phasor notation, **I** A, and **V** V, or:

**V****I** or **I****V** (8.3.10)

(Figure 8.3.2a). The magnitude of **V** is (1/*ωC)* times that of **I**, and the phase angle of **V** is -90° (Figure 8.3.2a).



According to the interpretation of complex sinusoidal excitation, a current

*Im*cos*ωt* A in a capacitor produces a voltage (*Im*/*ωC*)cos(*ωt* – *π*/2) = (*Im*/*ωC*)sin*ωt* across the capacitor, and a current *Im*sin*ωt* A produces a voltage (*Im*/*ωC*)sin(*ωt* – *π*/2) = (*Im*/*ωC*)cos*ωt*. The voltage *lags* the current by 90°, or the current leads the voltage by 90° (Figure 8.3.2). This can be ascertained by comparing the time of occurrence of the positive peaks of the waveforms of *v* and *i* in Figure 8.3.2b. Since the peak of *v* occurs later in time than the peak of *i* by a quarter of a period, *v* lags *i* by 90°.

The instantaneous power *p* absorbed by the capacitor is:

 (8.3.11)

and,  (8.3.12)

*p is* plotted in Figure 8.3.2b. *P* = 0, because an ideal capacitor does not dissipate power. The power absorbed (*p* > 0) and stored as electric energy during a period is equal and opposite to the power delivered (*p* < 0), when the stored energy is returned to the supply over the period.

***Phasor Relations for an Inductor***

If the inductor current is  A, the inductor voltage is:

 (8.3.13)

In phasor notation, **I** A, and **V** V, or:

**V****I**, or **I****V** (8.3.14)

The magnitude of **V** is *ωL* times that of **I**, and the phase angle of **V** is 90° (Figure 8.3.3a).



According to the interpretation of complex sinusoidal excitation, a current *Im*cos*ωt* A in the inductor produces a voltage *ωLIm*cos(*ωt* + 90°) = -*ωLIm*sin*ωt* across the inductor, and a current *Im*sin*ωt* A produces a voltage *ωLIm*sin(*ωt* – 90°) = -*ωLIm*cos*ωt*. The voltage *leads* the current by 90°, or the current *lags* the voltage by 90° (Figure 8.3.2(b)). This may be ascertained by comparing the time of occurrence of the positive peaks of the waveforms of *v* and *i.*

Since the peak of *v* occurs earlier in time than the peak of *i* by a quarter of a period, *v* leads *i* by 90°.

The instantaneous power *p* absorbed by the inductor is:

 (8.3.15)

and,  (8.3.16)

*p,* plotted in Figure 8.3.3b. *P* = 0, because an ideal inductor does not dissipate power. The power absorbed (*p* > 0) and stored as magnetic energy during a period is equal and opposite to the power delivered (*p* < 0), when the stored energy is returned to the supply over the period.

From the preceding discussion, the *v-i* relations of capacitors and inductors in the time domain and in phasor notation are as follows:

For a capacitor , and  (8.3.17)

For an inductor , and  (8.3.18)

It is seen that:

**Concept *In phasor notation, differentiation in time is expressed as multiplication by jω, and integration in time is expressed as division by jω. Thus, differential and integral relations are transformed to algebraic relations in jω for steady-state sinusoidal analysis ONLY.***

This concept in fact underlies the usefulness of phasor notation for steady-state sinusoidal analysis.

Another important observation is that *v* and *i* are in phase for an ideal resistor but are in phase quadrature for ideal capacitors and inductors. This is because an ideal resistor only dissipates power. It does not store energy that must be returned later to the supply. If no power is delivered by an ideal resistor, *p* is never negative, which means that *v* and *i* are in phase.

On the other hand, ideal capacitors and inductors do not dissipate power, which means that the average power *P* over a period is zero. For this to be the case, *v* and *i* must be in phase quadrature, which means that one is a cosine function and the other is a sine function, so that their product averages to zero over a period.

**Concept *The sinusoidal voltage and current for an ideal resistor are in phase because such a resistor is purely dissipative. They are in phase quadrature for ideal energy storage elements because these elements are nondissipative.***

**Exercise 8.3.1**

The voltage applied to a  resistor is peak, the frequency being 50 Hz. Determine: (a) the expression for the current in the time

domain, assuming the voltage is a cosine function; and (b) the power dissipated in the resistor.

Ans. (a) ; (b) 320 W.

**Exercise 8.3.2**

The current through a series combination of a 10 mH inductor and a  capacitor is  rms. Assuming the frequency is 200 Hz and the time variation is a sine function, determine the expression in the time domain of the voltage across: (a) inductor; and (b) the capacitor.

Ans. (a) 60*π*sin(400*πt +* 15°) mV rms; (b) (750/*π*) sin(400*πt –*165°) mV rms

**8.4 Impedance and Reactance**



Let **V** =  be a voltage phasor and **I** =  be a related current phasor, representing, for example, the voltage and current between any two nodes in a circuit (Figure 8.4.1), and let . Then:

***Definition Impedance Z is the ratio of the voltage phasor Vm∠θ v between any two nodes in a circuit to the current phasor******Im∠θ I flowing between these nodes in the direction of a voltage drop.***

 = *Z*  (8.4.1)

(Figure 8.4.2a). When **V** is in volts and **I** is in amperes *Z* is in ohms. Since *Z* is in general complex, it can be expressed as:

*Z* = *R* + *jX* (8.4.2)

(Figure 8.4.2b). It should be emphasized that although *Z* is in general complex, it is not a phasor, because it is not a complex sinusoidal function of time in which the time variation has been suppressed. In fact, the exponential time variation cancels out in Equation 8.4.1.



In the case of an ideal resistor, **V** and **I** are in phase (Equation 8.3.1), which means that

*Z* is real and equal to *R*. It follows that *R* in Equation 8.4.2 is the usual resistance that we have considered in previous chapters, and that for an ideal resistor, *X* = 0.

*X*, the imaginary part of *Z*, is the **reactance** and its unit is the ohm, just like resistance and impedance.

For an ideal capacitor, *Z* =  so that *X =* . For an ideal inductor, Equation 8.3.9 gives , so that . For both ideal energy storage elements, *R* = 0, since these elements are nondissipative.

It should be emphasized that when *X* = 0, **V** and **I** are in phase. When *X* ≠ 0, *Z* is complex and **V** and **I** are out of phase. Hence:

**Concept *In a purely resistive circuit, all sinusoidal voltages and currents are in phase and their magnitudes do not depend on frequency. Energy-storage storage elements possess reactance, which causes frequency-dependent phase differences between sinusoidal voltages and currents in the circuit and, in addition, makes the amplitudes of these sinusoidal voltages and currents depend on frequency.***

The following should be noted concerning reactance:

1. In any expression that involves impedance, reactance is always multiplied by *j*. For an ideal capacitor, *X* = -1/*ωC* and *Z* = -*j*/*ωC.* For an ideal inductor, *X* = *ωL* and *Z* = *jωL.*
2. Since *j* appears in the **V**-**I** relations as a result of differentiating or integrating expressions involving *ejωt*, *j* is always associated with *ω* in the expression for impedance. Hence, reactance is always a function of frequency.

The reciprocal of impedance is the **admittance** *Y*:

 (8.4.3)



where *B* is the **susceptance**. Like *G*, *B* and *Y* are in siemens.

For an ideal resistor, **I**/**V** = G (Equation 8.3.1), so that *B* = 0, and *G* is the usual conductance. For an ideal capacitor, *Y* = **I**/**V** = *jωC* (Equation 8.3.7), so that *G* = 0 and *B* = *jωC*. For an ideal inductor, *Y* = **I**/**V** = 1/*jωL* =`-1/ (Equation 8.3.9), so that *G* = 0, and *B* = -1/*ωL*. Table 8.4.1 lists the circuit properties of the three circuit elements. It is seen that under dc conditions *ω* = 0, which makes the reactance of a capacitor infinite and its susceptance zero. This means that when the voltage across the capacitor is dc, the current is zero. The capacitor behaves as an open circuit, as argued in Section 7.1, Chapter 7. Similarly, when *ω* = 0, the reactance of an inductor

is zero and its susceptance is infinite. This means that when the inductor current is dc, the voltage across the inductor is zero. The inductor behaves as a short circuit, as argued in Section 7.2, Chapter 7.



Note that whereas *G* = 1/*R* for an ideal resistor, *B* = -1/*X* for an ideal capacitor or inductor. This is because conversion from reactance to susceptance *must proceed through impedance and admittance*, which introduces a minus sign in the reciprocal of the imaginary component. Thus, Z = *jX*, *Y* = 1*/Z* =1/*jX* = -*j*/*X*. But Y = *jB*, so that *B* = -1/*X*.

**Exercise 8.4.1**

Find the reactance and susceptance of (a) the inductor and (b) the capacitor of Exercise 8.3.2.

Ans. (a) Inductor: Ω, S; (b) Capacitor: -50/*π* Ω, *π* /50 S.

Example 8.4.1

Given an impedance *Zs* = *Rs* + *jωLs*, represented as *Rs* in series with *Ls*. It is required to determine *Rp* and *Lp* of the equivalent parallel combination (Figure 8.4.3).



**S*olution*:** Since the two circuits are equivalent between terminals ‘ab’, they must have the same **V** and **I** at these terminals, so that the ratio **V**/**I** is the same. This means that *Zs* = *Zp* = 1/*Yp*. It follows that: . Rationalizing this fraction by multiplying the numerator and denominator by the complex conjugate of the denominator, so as to make the denominator real,

 (8.4.4)

Equating real and imaginary parts,

 and  (8.4.5)

From Table 8.4.1, *Gp* = 1/*Rp* and *Bp* = -1/*ωLp*. It follows that:

 (8.4.6)

and,  (8.4.7)

It should be noted that if, as is customarily the case, *Rs* and *Ls* are constants, such as 50 Ω and 10 mH, *Rp* and *Lp* are frequency dependent. They can only be considered constant at a specified frequency.

It is instructive to verify some limiting cases:

1. *Rs* = 0, as for an ideal inductor. This makes *Rp* → ∞, that is an open circuit, and *Lp* = *Ls*. The same pure inductance appears between terminals ‘ab’ in both cases.
2. *Ls* = 0, as for an ideal resistor. This makes *Lp* → ∞, that is an open circuit, and *Rp* = *Rs*. The same pure resistance appears between terminals ‘ab’ in both cases.
3. *ω* = 0, that is, dc conditions. Since *Ls* is assumed constant, *ωLs* = 0, which makes *Rs* = *Rp* in Equation 8.4.6 and *Lp* → ∞ in Equation 8.4.7. It follows that *Zp* = *Rp* = *Rs* = *Zs*.
4. *ω* → ∞, which makes *ωLs* → ∞ and *Zs* → ∞. Under these conditions, *Rp* → ∞, and *ωLp* = *ωLs* and tends to infinity as well. This makes *Zp* → ∞, so that terminals ‘ab’ are open-circuited in both cases.

**Problem-Solving Tip**

* Conversion of reactance, or reactance combined with resistance, to susceptance, or susceptance combined with conductance, must proceed through the intermediate step of impedance and admittance.

**8.5 Governing Equations**

The stated objective of sinusoidal steady-state analysis is to emulate dc analysis. In preceding chapters, circuit analysis under dc conditions was shown to be based on KCL, KVL, and Ohm’s law. In phasor notation for voltages and currents, which is derived for the sinusoidal steady state, time does not explicitly appear in the expressions for voltages and currents, just as in the dc case, because a steady state is assumed, Impedance and its reciprocal, admittance, relate phasor voltages and

currents of circuit elements in the same way as Ohm’s law relates dc voltage and currents of resistors. It remains to show, formally, that KCL and KVL apply in phasor notation.

If a current *i*1 enters a node and currents *i*2 and *i*3 leave this node, at any instant of time, KCL gives:

*i*1 = *i*2 + *i*3 (8.5.1)

where *i*1, *i*2,and *i*3 are instantaneous values. If these currents vary sinusoidally with time, then at any time *t*, Equation 8.5.1 can be expressed as:

*Im*1cos(*ωt* + *θ*1) = *Im*2cos(*ωt* + *θ*2) + *Im*3cos(*ωt* + *θ*3) (8.5.2)

where the amplitudes and phases of the currents are such that Equation 8.5.2 is satisfied. Since Equation 8.5.2 must be satisfied at any time *t*, it is satisfied a quarter of a period later, that is:

*Im*1cos(*ωt* + *θ*1 + *π*/2) = *Im*2cos(*ωt* + *θ*2 + *π*/2) + *Im*3cos(*ωt* + *θ*3 + *π*/2) (8.5.3)

*Im*1sin(*ωt* + *θ*1) = *Im*2sin(*ωt* + *θ*2) + *Im*3sin(*ωt* + *θ*3) (8.5.4)

Adding Equation 8.5.2 to Equation 8.5.4 multiplied by *j* expresses the currents as complex sinusoidal functions:

 (8.5.5)

Dropping the time variation leads to phasor notation:

**I1** = **I2** + **I3** (8.5.6)

It is seen that KCL applies to phasor currents. An exactly analogous arguments shows that KVL applies to phasor voltages.

Since all the concepts, theorems, and procedures discussed for the dc case are based on KCL, KVL, and Ohm’s law, and since the counterparts of these relations are satisfied in the sinusoidal steady state using phasors and impedances, the inevitable conclusion is that all the circuit relations, concepts, etc. that have been derived or applied under dc conditions carry over directly to the sinusoidal steady state.

**Concept *All circuit relations, concepts, theorems, and procedures that apply to resistive circuits under dc conditions apply to the sinusoidal steady-state, with voltages and currents represented as phasors and impedances of circuit elements replacing resistance.***

Specifically, this applies to all the circuit equivalence relations of Chapter 3, the circuit theorems of Chapter 4, the circuit simplification procedures of Chapter 5, and the circuit equations of Chapter 6. In fact, it can be asserted that the dc state can be considered as a special case of the sinusoidal steady state with the frequency set to zero, so that inductors are replaced by short circuits and capacitors by open

circuits.

Example 8.5.2 illustrates voltage and current division using impedances.

**Exercise 8.5.1**

Given an inductor of 50 mH inductance and 10 Ω resistance. Determine: (a) the impedance of the inductor, and (b) the equivalent parallel resistance and inductance (Figure 8.4.3b), assuming a frequency of 1 krad/s.

Ans. (a) 10 + *j*0.05*ω* Ω; (b) *Rp* = 260 Ω, *Lp* = 52 mH.

**Exercise 8.5.2**

A 50 Ω resistor is connected in parallel with a 20 μF capacitor. Determine the resistance and reactance of the parallel combination at a frequency of 1 krad/s.

Ans. 25 Ω, -j25Ω.

**Example 8.5.1**

Given the circuit of Figure 8.5.1, it is required to determine **VL***,* **IL**, **I1**, and**I2**, assuming *ω* = 100 rad/s.



**S*olution*:** The reactance of the capacitor is  Ω; *Z2* = 10 – *j*10 Ω. The reactance of the inductor is   Ω; *ZL* = 20 + *j*20 Ω. *Z*2 in parallel with *ZL* is:   Ω. Hence, **I1**  A. From voltage division, ×**VSRC**  V.

From current division: **IL****I1** =

A. From KCL, **I2** = **I1** – **IL** =  A.

**Simulation:** The circuit is entered as Figure 8.5.2. For the sinusoidal steady-state, source VAC is used at a single frequency. A magnitude is entered, and a phase, if required, can be set in the Property Editor spreadsheet. Note that the value entered for VAC is considered by PSpice as an rms value for determining power values. But for determining voltages or currents, the value entered could be interpreted either as a peak value or an rms value.



Printers from the SPECIAL library can be used for measuring voltages and currents. A current printer is inserted in series with one or more circuit elements through which the current in question passes. The positive direction of current is that entering the unmarked terminal of the current printer and leaving the terminal marked with a minus sign. A one-terminal voltage printer is used for measuring voltages with respect to ground, and a two-terminal voltage printer is used for measuring voltages between two nodes. For AC measurements, a Y must be entered under AC in the Property Editor spreadsheet of each printer. The printer reading is in rectangular coordinates if a Y is entered under REAL and IMAG, and the reading is in polar coordinates if a Y is entered under MAG and PHASE.

In the simulation profile, AC Sweep/Noise is chosen under Analysis type. For sinusoidal steady-state analysis, a single frequency is used. Under AC Sweep Type, either Linear or logarithmic may be selected, The frequency is entered in Hz and is Hz in this case. The same frequency is entered for Start Frequency and End Frequency, and a 1 is entered for Total Points. After the simulation is run, the printer readings are available at the end of Simulation Output File. This file can be viewed by selecting View/Output File at the top of the SCHEMATIC1 page, or by double clicking on the third icon from the top at the left margin of the SCHEMATIC1 page. This icon is labeled View Simulation Output File. The printer readings are as

follows:

FREQ IM(V\_PRINT1) IP(V\_PRINT1)

1.592E+01 4.903E-01 1.131E+01

FREQ IM(V\_PRINT2) IP(V\_PRINT2)

1.592E+01 4.385E-01 3.788E+01

FREQ IM(V\_PRINT3) IP(V\_PRINT3)

1.592E+01 2.193E-01 -5.213E+01

FREQ VM(A) VP(A)

1.592E+01 6.202E+00 -7.125E+00

The current printers are identified by their number, and the voltage printer by the label of the node to which the printer is connected. If the node is not labeled, the reference is to the pin number of the node. This number can be read by pointing the cursor at the node in the circuit entered. IM and IP refer to current magnitude and current phase; similarly for VM and VP. A value such as 4.903E-01 denotes 4.903×10-1. It is seen that printer values agree with those calculated.

An alternative to using printers for reading values of currents and voltages is to use the Evaluate Measurement feature of PSpice. In the SCHEMATIC1 page, select Trace/Evaluate Measurement. Two windows are displayed. Under Functions or Macros in the right-hand window, choose Analog Operators and Functions. To read magnitudes, select the voltage or current required from the left-hand window labeled Simulation Output Variables. To read phase angles, select P(), then select the variable required. For example, to read the voltage of the node labeled ‘a’, V(a) and P(V(a)) are selected. The values 6.20174 and -7.12502 are displayed in a window in the SCHEMATIC1 page. To read the values of **I1**, **I2**, and **IL**, the variables are -I(V1), I(C1), and I(L1), respectively.



**Exercise 8.5.3**

Determine

the impedance

and admittance

between terminals

‘ab’ in Figure 8.5.3

at *ω* = 1 krad/s.

Ans. *j*25 Ω; -*j*40 mS.

**8.6 Representation in the Frequency Domain**



Let us return to Figure 8.2.1 that we started with in order to illustrate the general procedure for analyzing the sinusoidal steady state in the same manner as a dc steady state. The source *Im*cos*ωt* is converted to a phasor **ISRC** = *Im*∠0, and *R* and *L* are expressed as impedances, *R* and *jωL*, respectively, in series with the source. The circuit in terms of phasors and impedances or admittances is now said to be in the *frequency domain*, as illustrated in Figure 8.6.1.

A direct application of KVL gives:

**V** = *R***ISRC** + *jωL***ISRC** = (*R* + *jωL*)**ISRC** = *Z***ISRC** (8.6.1)

This is an algebraic equation involving the phasors and the complex impedance *Z* = *R* + *jωL.* The relation **V** = *Z***ISRC** is exactly analogous to ohm’s law in the dc case, bearing in mind that whereas dc voltages and currents have only values, ac voltages and currents have magnitude and phase.

Expressing Z in polar coordinates, Equation 8.6.1 becomes:

**V** = *Im*∠0 =  (8.6.2)

**V** can now be converted back to the time domain as a cosine function, since *iSRC* is a cosine function. This gives:

 (8.6.3)

which is identical with Equation 8.2.4.

The procedure for deriving the sinusoidal steady-state response can be generalized and summarized as follows:

1. *The sinusoidal excitations are expressed as phasors. A single excitation Ymcos(ωt + θ), or Ymsin(ωt + θ), can be expressed as Ym∠θ. In the case of several excitations, some of which are cosine functions and some are sine functions, they should all be converted to either cosine or sine functions before expressing them as phasors.*
2. *Inductance and capacitance are expressed as impedance or admittance, as appropriate.*
3. *The circuit is now in the frequency domain and can be analyzed by any of the methods discussed in previous chapters for the dc case.*
4. *After obtaining the desired response, this response can be converted back to the time domain as a cosine function, if the time functions that were originally expressed as phasors were cosine functions, or converted as a sine function if*

*the time functions that were originally expressed as phasors were sine functions,*

**Example 8.6.1**

It is required to determine Norton’s equivalent circuit seen by the 20 Ω resistor between terminals ‘ab’ in Figure 8.6.2, assuming *vSRC*1 = 200sin(5×104*t*) V and *vSRC*2 = 100cos(5×104*t*) V.



**Solution:** The circuit is first represented in the frequency domain. Since one source voltage is a cosine function and the other source voltage is a sine function, they should both be expressed as sine functions or cosine functions. As a cosine function, *vSRC*1 = 200cos(5×104*t* – 90) V. As phasors, **VSRC1** = 200∠-90° = -*j*200 V, and **VSRC2** = 100∠0 = 100 V. *ω* = 5×104 rad/s, *jωL* = *j*5×104×0.4×10-3 = *j*20 Ω. ****= -*j*20 Ω. The circuit in the frequency domain is shown in Figure 8.6.3 with terminals ‘ab’ short circuited.



**ISC** can be determined by superposition. With **VSRC1** applied alone, and **VSRC2** replaced by a short circuit (Figure 8.6.4a), the capacitive reactance is short circuited, and the current through it is zero. Hence, **ISC1** =



-*j*200/*j*20 = -10 A. With **VSRC2** applied alone and **VSRC1** replaced by a short circuit (Figure 8.6.4b), the inductive reactance is short circuited, and the current through

It is zero. Hence, **ISC2** = 100/(-*j*20) = *j*5 A. It follows that **ISC** = **IN** = -10 + *j*5 = 5(-2 + *j*) A.

If both sources are replaced by short circuits, the *j*20 Ω and the -*j*20 Ω appear in parallel across terminals ‘ab’ (Figure 8.6.5a). The 20 Ω resistor is short circuited. The impedance between terminals ‘ab’ is *ZTh* = (*j*20)(-*j*20)/(*j*20 – *j*20) → ∞. Alternatively the admittance of the inductor is -*j*/20 S, and the admittance of the capacitor is *j*/20 S. The admittance *YTh* between terminals ‘ab’ is zero. NEC is an ideal current source 5(-2 + *j*) A. Since *ZTh* → ∞, TEC does not exist.



**Simulation:** Unlike the dc case, TEC or NEC cannot be obtained in a single simulation, because ac quantities have both magnitude and phase. The circuit for determining NEC is entered as in Figure 8.6.6a. Because the current printer has zero resistance, a voltage-source-inductor loop is formed that is not allowed in PSpice. This loop can be broken by inserting a 1u resistance as shown, which is too small to affect the result. In the simulation profile, *f* = 5×104/2*π* = 7957.75 Hz is entered for Start Frequency and End Frequency, and a 1 for Total Points. After the simulation is run, the printer readings are as follows:

FREQ IR(V\_PRINT1) II(V\_PRINT1)

7.958E+03 -1.000E+01 5.000E+00

7.958E+03 2.500E-09 3.576E-08

where the first reading is that of **ISC** in Figure 8.6.6a and the second reading is that of the current of the test voltage source in Figure 8.6.6b. IR is the real part of the current and II is the imaginary part. The first reading agrees with the calculated value, and the second reading is insignificantly small, which is interpreted as zero.

**Example 8.6.2**

It is required to determine *vO* in Figure 8.6.7 assuming *vSRC* = 100cos(103 – 30°) V.



**Solution:** **VSRC** = 100∠-30° V, *jωL* = *j*103×20×10-3 = *j*20 Ω, = ****

****Ω.

The circuit will be analyzed by the node-voltage method. The voltage source in series with the 40 Ω resistance is transformed to a current source **ISRC** = 2.5∠-30° A in parallel with 40 Ω. In rectangular coordinates, **ISRC** = 2.5(cos30° – *j*sin30°) = 1.25(. The circuit in the frequency domain becomes



as shown in Figure 8.6.8. **IO** = **V2**/20.

Bearing in mind that the coefficients of the node voltages are admittances when using phasor analysis, the node-voltage equation

for the node on the left is:

**V1** –**V2** = **V2** , or, **V1** **V2** = .

The node-voltage equation for the node on the right is: –**V1** + **V2** = 0. Solving for **V2** gives **V2** = 5.021 + *j*27.65 V. Hence **VO** = **V2** = 12.1 + *j*3.52 V.



**Simulation:** The circuit is entered as in Figure 8.6.9. Note that as an alternative to reversing the direction of **IO** through the dependent source, the sign of the gain is made negative. In the

simulation profile, *f* = 103/2*π* = 1591.55 Hz is entered for Start Frequency and End Frequency, and a 1 for Total Points. After the simulation is run, the printer readings are as follows:

FREQ VR(A) VI(A)

1.592E+03 12.06E+00 3.522E+00

**Example 8.6.3**

It required to determine **IO** in Figure 8.6.10a.

**Solution:** The circuit is given in the frequency domain. **IO** will be determined by the mesh-current



method, the mesh-current assignments being as in Figure 8.6.10b. Because of the nontransformable

current source in meshes 2 and 3, a voltage **VY** is arbitrarily assigned across this source to allow writing the mesh-current equations for these meshes. The mesh-current equations are:

(2 + *j*2)**I1** – *j*2**I2** – 2**I3** = *j*10; -*j*2**I1** + (4 + *j*2)**I2** = **Vy**; -2**I1** + (2 – *j*4)**I3** = -**Vy**. Adding the last two equations to eliminate **VY**, the resulting equation is: -(2 + *j*2)**I1** + (4 + *j*2)**I2** + (2 – *j*4)**I3** = 0. The current source gives: **I2** – **I3** = 5. From the solution of the equations, **I1** = 1.25 + *j*3.75 and **I3** = -3.75 – *j*1.25, so that **IO** = **I1** – **I3** = 5 + *j*5 A.

**Simulation:** The circuit is entered as in Figure 8.6.11. Since the reactances are specified in the circuit, but inductance and capacitance values have to be entered in PSpice, any convenient value of *ω* can be assumed and *L* and *C* calculated accordingly. Thus, if *ω* = 1 rad/s, then *L* = *X*/*ω* = 2 H, and C = -1/*ωX* = 0.25 F, as entered in the circuit. In the simulation profile, *f* = 1/2*π* = 0.159155 Hz is entered for Start Frequency and End Frequency, and a 1 for Total Points. After the simulation is run,



the printer readings are as follows:

FREQ IR(V\_PRINT1) II(V\_PRINT1)

1.592E-01 5.000E+00 5.000E+00

**8.7 Phasor Diagrams\***

Phasor diagrams showing various voltage and current phasors in a given circuit are useful for illustrating the interrelations between the various variables involved, particularly when some circuit variable is varied, as illustrated by the following example.

### Example 8.7.1

Given the circuit of Figure 8.7.1. It is required to determine: (a) the value of *R* at which *vO* is 90° out of phase with respect to *vI*, and (b) how *vO* changes as *R* is varied from 0 to infinity, assuming that no current is drawn at the output.



**Solution:** -*j*/*ωC* = -*j*/(103×10-6) = -*j* kΩ. The circuit configuration of Figure 8.7.1 is referred to as a lattice configuration. Because it contains a crossover connection, it is not easy to visualize circuit behavior. The crossover connection can be removed, however, by redrawing the circuit. This is conveniently done by labeling the nodes and tracing the circuit in a given direction. Thus, starting with node ‘a’ in Figure 8.7.1, the upper 1 k Ω resistor is drawn between this node and node ‘b’ (Figure 8.7.2a), then going through the original circuit and passing through the second 1 k Ω resistor, this resistor is drawn between node ‘b’ and node ‘c’, followed by *R* to node ‘d’, and finally through *C* back to node ‘a’. The resulting circuit is a bridge circuit, as



shown in Figure 8.7.2a in the frequency domain, with **VI** applied between nodes ‘a’ and ‘c’, and **VO** taken between nodes ‘b’ and ‘d’.

(a) **VI** = *Vm*∠0. By voltage division, *Vab* = *Vm*/2 and *Vbc* = *Vm*/2. From KVL around the mesh ‘cbd’, *Vm*/2 – **VO** – *Vdc* = 0. By voltage division, *Vdc* = *VmR*/(*R – j*) = *VmR*/(*R – j*).It follows that:

 (8.7.1)

The magnitude of **VO** is given by:

 (8.7.2)

It is seen that |**VO**| is *Vm*/2, independently of *R*. To ascertain the correct value of the phase angle, the numerator and denominator in Equation 8.7.1 are drawn as phasors in Figure 8.7.3. It is seen that:



∠**VO** = -*π* + *θ* – (-θ) = -*π* + 2*θ* (8.7.3)

where *θ* = tan-1(1/*R*) and has a positive value. ∠**VO** is negative, so that **VO** lags **VI**. To have **VO** 90° out of phase with **VI**, ∠**VO** = -90. Substituting in Equation 8.7.3, *θ* = tan-1(1/*R*) = 45°. This gives *R* = 1 kΩ and ∠**VO** =

-45°.

When *R* = 0, tan-1(1/*R*) = 90°, and ∠**VO** = -180° + 180° = 0. In Figure 8.7.4a. Node ‘d’ is connected to ‘c”, which makes **VO** = *Vbc* = *Vm*/2, so that ∠**VO** = 0. When *R* → ∞, *θ* = tan-1(1/*R*) = 0. In Figure 8.7.4, **I** = 0, which makes nodes ‘a’ and ‘d’ are at the same voltage, → **VO** = *Vba* = - *Vm*/2, and ∠**VO** = 180°.



(b) From KVL in Figure 8.7.2b, **VO** = *R***I** + *jX***I** = *R***I** – *j***I**. The phasor *Vm*∠0 is drawn in Figure 8.7.5 as OP. The phasor -*j***I** lags the phasor *R***I** by 90, yet their sum must always be *Vm*∠0. It follows that the point Q joining these two phasors lies on the perimeter of a semicircle of diameter *Vm* (Figure 8.7.5). If a phasor -*R***I** is drawn from

point T at the tip of the phasor (*Vm*/*2*)∠0 then **VO** = (*Vm*/2) – *R***I** is the phasor from the origin O to S at the tip of -*R***I**.



When *R =* 0, phasors *R***I** and -*R***I** are zero. Q coincides with O, and S coincides with T, so that **VO** = *Vm*/2, as deduced from figure 8.7.4a. When *R* → ∞, **I** = 0 (Figure 8.7.3b) so that Q coincides with P and S coincides with U. **VO** = -*Vm*/2, again as deduced from figure 8.7.4b. As *Rx* increases from zero, Q moves clockwise around the perimeter of its semicircle. Likewise, Smoves clockwise around the perimeter of a semicircle, and *β*, the phase angle of *VO*, varies between 0 and -180°.

It is seen from the geometry that, where =  Substituting, gives: . When *R* = 1, , and *β* = -45°.

The circuit may be used to shift the phase of the output with respect to the input, without altering the magnitude.

# Summary of Main Concepts and Results

* The sinusoidal function is only periodic function that has only a single frequency.
* When a sinusoidal excitation is applied to an LTI circuit, all the currents and voltages in the circuit are sinusoidal functions of the same frequency as the excitation, but which generally differ in amplitude and phase angle.
* When a complex sinusoidal excitation *Kejωt* is applied to an LTI circuit, the response is a complex sinusoidal function whose real part is the response to the real part of the excitation, *Kcosωt,* applied alone, and whose imaginary part is the response to the imaginary part of the excitation, *Ksinωt,* applied alone*.*
* A phasor is a quantity such as  representing a complex sinusoidal function of time, but with the time variation suppressed. They have the same geometric properties as vectors but have real and imaginary components.
* A current of rms value *I*rms, or a voltage of rms value *V*rms, dissipates the same power in a given resistor as a DC current, or a DC voltage, of the same value.
* In phasor notation, differentiation in time is replaced by multiplication by *jω* and integration in time is replaced by division by *jω*. Thus, differential and integral relations are transformed to algebraic relations in *jω* for steady-state sinusoidal analysis ONLY.
* The sinusoidal voltage and current for an ideal resistor are in phase because such a resistor is purely dissipative. They are in phase quadrature for ideal energy storage elements because these elements are nondissipative.
* Impedance Z is the ratio of the voltage phasor  to the related current phasor . The real part of impedance is resistance and its imaginary part is reactance.
* Reactance is due to energy storage elements. Whereas sinusoidal voltages and currents are all in phase in a purely resistive circuit, energy storage elements introduce phase differences between sinusoidal voltages and currents.
* All circuit relations, concepts, theorems, and procedures that apply to resistive circuits under DC conditions apply to the sinusoidal steady-state, with voltages and currents represented as phasors and impedances of circuit elements replacing resistance.

**Problem-Solving Tips**

1. Complex quantities are conveniently added or subtracted in rectangular form, and are conveniently multiplied or divided in polar form.
2. Conversion of reactance, or reactance combined with resistance, to susceptance, or susceptance combined with conductance, must proceed through the intermediate step of impedance and admittance.

**Exercises and Problems**

**8.1 Sinusoidal Functions and Phasors**

**P8.1.1** Given *y* = 15cos(200*πt* + 30°). Determine: (a) the duration of 10 cycles; (b) *t* at which the first zero of the 11th cycle occurs.



Ans. (a) 100 ms; (b) 101.67 ms.

**P8.1.2** Given the sinusoidal time function  of

Figure P8.1.2. Express  as a function

of time and as a phasor.

Ans. 10cos(200*π*t + 72.54°) V, **V** =10∠72.54° V.

**P8.1.3** A current *i* = *2.5*cos(*ωt* + 30°) flows through an impedance (5 – *j*5) Ω. Determine the rms phasor voltage across the impedance.

Ans.  V

**P8.1.4** Given a phasor **A** = *A*∠*α*. Express the phasor ‘Ob’ in terms of **A** and *j*, assuming that the phasors ‘Oa’ and ‘ab’ have a magnitude *A*.



Ans. *j*(1 + *j*)**A**.

**P8.1.5** Given **A** **B** and **C** Determine the phasors resulting from the following operations: (a) **A** + **B** + **C**; (b) **A** – **B** + **C**; (c) **A** + **B** – **C** and (d) **A** – **B** – **C**.Express the result in rectangular and polar forms.

Ans. (a) 3.19 + *j*16.37, 16.68∠78.96°; (b) 23.19 – *j*18.27, 29.52∠-38.22°; (c) -3.88 + *j*23.44, 23.76∠99.39°; (d) 16.12 – *j*11.20, 19.63∠-34.78°.

**P8.1.6** Given **A** = 3 + *j*5, **B** = 10 – *j*8, and **C** = *j*12. Determine the phasors resulting from the following operations: (a) **A**\***B**\***C**; (b) (**A**\***B**)/**C**; (c) (**A**/**B**)\***C**; and (d) **A**/**B**/**C**. Express the result in rectangular and polar forms.

Ans. (a) -312 + *j*840, 896.1∠110.4°; (b) 2.1667 – *j*5.8333, 6.223∠-69.62°;

(c) -5.4146 – *j*0.7317, 6.223∠-172.3°; (d) 0.0376 + *j*0.0051, 0.0379∠7.69°.

**P8.1.7** Given **A** = 5 + *j*10. Determine the phasor that is **A** raised to the fourth power.

Ans. -4375 –*j*15,000, 15,625∠-106.3°.

**P8.1.8** Given **A** = 24 +*j*32. Determine the phasor that is the cube root of **A**.

Ans. 3.42∠17.71°, 3.42∠137.71°, 3.42∠257.71°.

**P8.1.9** Using phasors, determine the steady-state *y* that satisfies the differential equation:



Express *y* as a cosine time function. (Hint:

Ans. 0.472cos(4*t* – 146.2°).

**P8.1.10** Verify conservation of power in Example 8.5.1

**8.2 Impedance and Admittance**

**P8.2.1** Given an impedance 0.1(4 + *j*3) Ω. Determine the susceptance.

Ans. *B* = -1.2 S.

**P8.2.2** A susceptance of -1 S is connected in series with an admittance (3 + *j*4) S. Determine the reactance of the series combination.

Ans. 0.84 Ω.

**P8.2.3** *v* in Figure P8.2.3 is the voltage between two terminals of a given circuit, and *i* is the current entering these terminals in the direction of a voltage drop *v*. Determine the impedance looking into these terminals.



Ans. 2∠-45° Ω.

**P8.2.4** Given a inductor and a  capacitor. (a) At what frequency is the impedance of the series combination zero? (b) At what frequency is the admittance of the series combination zero? (c) At what frequency is the admittance of the parallel combination zero? (d) At what frequency is the impedance of the parallel combination zero?

Ans. (a) and (c) (b) and (d) 0 and 

**P8.2.5** Determine **Vx** and **IL** in Figure P8.2.5 and the total power dissipated in the circuit, assuming the supply frequency is 



Ans.  V, 0.2 A; 1.5 W.



**P8.2.6** Determine **IL** and **VC** in Figure P8.2.6, assuming the supply frequency is 

Ans. 0.5 – *j*4 A, 200 – *j*20 V.



**P8.2.7** Determine *ω* at which the voltage across the 10 Ω resistor in Figure 8.2.7 is a maximum and specify this voltage.

Ans. *ω* = 2 krad/s, 2 V.



**P8.2.8** Determine *Zi*, the impedance looking into terminals ‘ab’ in Figure P8.2.8. (Hint: apply the source absorption theorem).

Ans. *j* S



**P8.2.9** Determine *Zi*, the impedance looking into terminals ‘ab’ in Figure P8.2.9, assuming σ = 1/(2 – *j*2) S.

Ans. *j*2 Ω.

**P8.2.10** All the inductances in Figure P8.2.10 are *j*10 Ω, all the resistances are 10 Ω, and all the capacitances are -*j*12 Ω. Determine the impedance between terminals ‘ab’.



Ans. 27 + *j*9 Ω.

**P8.2.11** Determine Thevenin’s impedance between terminals ‘ab’ in Figure 8.2.11.



Ans. 2 – *j*8 Ω.



**P8.2.12** Determine *Z* in Figure 8.2.12 so that the Thevenin’s impedance between terminals ‘ab’ is 1 Ω.

Ans. 0.8 − *j*1.4 Ω.

**P8.2.13** A coil has a resistance of 10 Ω and an inductance L. when connected to a 100 V rms, 60 Hz supply, the coil current is 5 A rms. Determine *L*.

Ans. 45.9 mH.



**P8.2.14** (a) Express the impedance looking into terminals ‘ab’ in Figure P8.2.14 in terms of *ω*; (b) determine *ω* so that *Zi* is purely resistive.

Ans. (a) ;

(b) 5.6 kHz.

**P8.2.15** A capacitor of impedance *ZC* is connected in parallel with a load of (300 + *j*450) Ω. Determine *ZC* so that the equivalent load is purely resistive.

Ans. -*j*650 Ω.



**P8.2.16** Determine Thevenin’s impedance looking into terminals ‘ab’ in Figure P8.2.16.

Ans. -*j*20 Ω.



**P8.2.17** Determine *Z*, given that **IS** = 27.9∠57.8° A.

Ans. 5∠-29.9° Ω.



**P8.2.18** Given *i* =2500*πt* – 45°) A and *i*1 = 2cos(2500π*t*) A in Figure P8.2.18. Determine: (a)  and  in the time and frequency domains; (b)  if composed of i) two series elements, or ii) two parallel elements.

Ans. (a) *v* = 145.02cos(2500*πt* – 57.87°) V, **V** = 145.02∠-57.87° V, *i*2 = 2.9cos (2500π*t* – 110.49°) A, **I2** = 2.9∠-110.49°;



(b) i) *R* = 15.43 Ω and *C* = 21.65 μF, ii) *G* = 0.0566 S and *C* = 2.75 μF.

**P8.2.19** Given *vSRC* =  V in Figure P8.2.19. Determine *iSRC* as a phasor in rectangular coordinates.

Ans. **ISRC** *=* -*j*2 mA.



**P8.2.20** Determine the frequency at which the impedance looking into terminals ‘ab’ is purely resistive, given *L* = 2.5 mH.

**8.3 General Sinusoidal Steady State Analysis**

**P8.3.1** Determine **IL** in Figure P8.3.1, assuming *ω* = 100 rad/s.



Ans. 1∠-90° A.



**P8.3.2** Determine **IC** in Figure P8.3.2, assuming *ω* = 2 krad/s.

Ans. 5∠135° A.

**P8.3.3** Determine *vL* in Figure P8.3.3



as a phasor and in the time domain.

Ans. **VL** =  V, *vL* = 1.89cos(*t* + 90°) V.



**P8.3.4** Determine *vO* in Figure P8.3.4, assuming *ω* = 104 rad/s.

Ans. 10.54cos(*ωt* – 63.43°) V.

**P8.3.5** Determine *C* in Figure P8.3.5 so that *vO* has the same magnitude as *vI* but lags it by 90°, assuming ω = 400 rad/s.



Ans.



**P8.3.6** Determine **Vx** in Figure P8.3.6.

Ans.  V.



**P8.3.7** Determine the power dissipated in *R*1 in Figure P8.3.7, assuming *ω* = 1 krad/s.

Ans. 0.1 W.



**P8.3.8** Determine **Ix** in Figure P8.3.8

Ans. -*j*3 A

**P8.3.9** Determine *R* in Figure P8.3.9, given **I** = 0.



Ans. 1 Ω.



**P8.3.10** Determine *Z* in

Figure P8.3.10.

Ans. 68 + *j*24 Ω.



**P8.3.11** |**I|** in Figure P8.3.11 remains the same irrespective of whether the switch is open or closed. Show that under these conditions 2*ω*2*LC* = 1.

**P8.3.12** Determine **Vab** in Figure P8.3.12, assuming all impedances are in ohms.



Ans. -741 + *j*494 V.

**P8.3.13** Determine **I1** and **I2** in Figure P8.3.13.



Ans. **I1** = -0.0517 − *j*0.621 A,

**I2** = -0.466 + *j*1.91 A



**P8.3.14** Determine **IS** in Figure P8.3.14.

Ans. 0.5 A.



**P8.3.15** Determine **IO** in Figure P8.3.15.

Ans. 11.57∠89.6

**P8.3.16** Determine **VO** in Figure P8.3.16

Ans. -9 + *j*10 V.



**P8.3.17** Determine: (a) *ix* in Figure P8.3.17, given *vSRC* = cos*t* and *iSRC* = sin2*t*; (b) the power dissipated in the resistor.



Ans. (a) 0.71cos(*t –* 45°) + 0.89sin(2t + 26.6°) A; 0.65 W.



**P8.3.18** Determine *iO* in Figure P8.3.18, given *vSRC* = 10cos(3000*t*) V.

Ans. -1 + 2 cos(3000*t*) V.

**P8.3.19** Determine **Vc** and **IL** in Figure P8.3.19.

Ans, **Vc** = -30 – *j*90 V, **IL** = 8 − *j*6 A.



**P3.1.20** Determine **Ix** in Figure P8.3.19.



Ans. -*j* A

**P8.3.21** Given *iSRC* = cos108*t* A. Determine: (a) **Vx** and **Ix**; (b) *vO*.



Ans. (a) **Vx** = 60 – *j*20 V, **Ix** = 0.2 + *j*0.6 A; (b) *vO* =  V.



**8.4 TEC and NEC**

**P8.4.1** Derive NEC looking into terminals ‘ab’ in Figure P8.4.1.

Ans. *j*3 A in parallel with *j*10/3 Ω.



**P8.4.2** Derive TEC looking into terminals ‘ab’ in Figure P8.4.2.

Ans. (-50 + *j*25) V in series with (5 + *j*10) Ω.



**P8.4.3** Derive TEC looking into terminals ‘ab’ in Figure P8.4.3.`

Ans. 57.3∠-55.0° V in series with (4.7 – *j*6.71) Ω.



**P8.4.4** Derive TEC looking into terminals ‘ab’ in Figure P8.4.4.

Ans. 5cos(10*t*) V in series with (0.1 + *j*1.8) Ω.

**P8.4.5** Derive NEC looking into terminals ‘ab’ in Figure P8.4.5.



Ans. 50∠0° A in parallel with (2 + *j*) Ω.

**P8.4.6** Derive TEC looking into terminals ‘ab’ in Figure P8.4.6.



Ans. 12 + *j*6 V in series with 20 Ω.



**P8.4.7** Derive TEC looking into terminals ‘ab’ in Figure P8.4.7.

Ans. *VTh* = - *j*15 V, *ZTh* = 0.



**P8.4.8** Determine *Z* so that **VO** is in phase with the source voltage.

Ans. *j*2.5 Ω.

**P5.4.9** Derive NEC looking into



terminals ‘ab’.

Ans. -0.1 A in parallel with

(4 – j3)/250 S.

**8.5 Node-Voltage and Mesh-Current Methods**

**P8.5.1** Determine  in Figure P8.5.1 using the mesh-current method, given *vSRC*1 = 10cos(104*t* + 45°) V and *vSRC*2 = 10cos(104*t* − 45°) V.



Ans. 0.133cos(*ωt* + 130.1°) A.

**P8.5.2** Determine **ISRC** and **VO** in Figure P8.5.2



using the node-voltage method.

Ans. **VO** = 13.4 + *j*11.7 V,

**ISRC** = 0.469 + *j*0.164 A.



**P8.5.3** Determine **VO** in Figure P8.5.3 using the mesh-current method.

Ans. **VO** = 13.98 − *j*2.851 V.

**P8.5.4** Determine  in Figure P8.5.4 using the node-voltage method.



Ans. **VO** = 4.88 – *j*20.0 V.



**P8.5.5** Determine  in Figure P8.5.5 using the mesh-current method.

Ans. **IO** = 5 + *j*5 A.



**P8.5.6** Determine *vc* in Figure P8.5.6 using the node-voltage method.

Ans. 12.1sin(2000t + 6.86°) V.



**P8.5.7** Determine  in Figure P8.5.7 using the mesh-current method.

Ans. **IO** = 5 − *j*9 A.

**P8.5.8** Determine  in Figure P8.5.8 using the node-voltage method.



Ans. **VO** = 1.818 V.